Log Stacks

Let $\mathcal{C}$ be a site (cat. + Groth-top.). A stack over $\mathcal{C}$ is a cat $p: S \to \mathcal{C}$ s.t.

1. $p: S \to \mathcal{C}$ is a fibred cat.
2. $\forall\, \mathcal{U} \in ob(\mathcal{C})$, $\forall\, x, y \in S_{\mathcal{U}}$, the presheaf $Hom(x,y)$ is a sheaf on the site $\mathcal{C}/_{\mathcal{U}}$ ('slice').
3. for any covering $\mathcal{U} = \{f_i : U_i \to U\}_{i \in I}$ in $\mathcal{C}$, any descent datum in S relative to $\mathcal{U}$ is effective.

See Stacks, tag 026F for links to all requisite defs.

We give Sch the etale top, & $LSch$, $LSch^{fs}$ the strict etale tops.

From Garnet's talk:

- $\mathcal{L}M_{g,n}$:= cat. of stable logcurves of type g,n
- $\mathcal{L}M_{g,n} \to \mathcal{L}Sch^{fs}$ is a stack over $\mathcal{L}Sch^{fs}$.
  $(X/S) \longmapsto S$
- the full subcat $M^{bas}_{g,n} \to \mathcal{L}M_{g,n} \to \mathcal{L}Sch^{fs} \to Sch$
  $(X, M_X) \longmapsto X$
  makes $M^{bas}_{g,n}$ into a stack over Sch.
- the nat. map $M^{bas}_{g,n} \to \overline{M}_{g,n}$ is an iso of stacks over Sch.
  $(X/_S, M_X) \longmapsto (X/_S)$
- ~~A log~~ let X be a stack over Sch. A log str. on X is a functor $X \to \mathcal{L}Sch$ s.t.

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & \mathcal{L}Sch \arrow[d] \\ & Sch \end{tikzcd}$$

commutes (can add fs)

- The map $M^{bas}_{g,n} \to \mathcal{L}Sch^{fs}$ above makes $M^{bas}_{g,n}$ a (fs) log stack
  (= stack$/_{Sch}$ + (fs) log str)
- The log str on $M^{bas}_{g,n}$ is the divisorial log str. from the (NC) boundary

Let (S, L) be a logstack (so $S/_{Sch}$ a stack, $L : S \to \mathcal{L}Sch$ with $S \searrow Sch \swarrow$). ②

We define a cat. S^{log} by

- objects: $\underbrace{\text{morphism } (X, M) \to (S, L) \text{ of logstacks}}$, where (X, M) is a log scheme.
  (i.e. map of stacks + nat. trans $X \xrightarrow{\varphi} S$, ~~…~~ $M \to \varphi^* L$, $\varphi^* L = L \circ \varphi$)
- morphism: a morphism of logsch making obv. Δ commute.

~~…~~ Write $\mathcal{M}_{g,n}^{log}$ for the above cat. assoc. to $\mathcal{M}_{g,n}^{bas}$.

We get a ~~map~~ functor

$$\mathcal{M}_{g,n}^{log} \xrightarrow{F} \mathcal{L}\mathcal{M}_{g,n}$$

$$(T \to \mathcal{M}_{g,n}^{bas}) \longmapsto \left(\begin{array}{c}\text{the universal curve} \\ \text{over } T \text{, with } \sout{\text{basic}} \text{ logs. str} \\ \text{pullback of basic logstr.}\end{array}\right)$$

Thm F is an equivalence (of fibred cat. over $\mathcal{L}Sch$).

"Pf." Give an inverse. Suppose $X'/_{T'}$ is a log curve of type (g, n), so an obj of $\mathcal{L}\mathcal{M}_{g,n}$.

Then on $\dfrac{X}{T}$, can put basic logstr.

Apply Gamet's prop (2-3 of Kato) to $X' = X'$, $S' = T'$, $X = X_{bas}$, $S = T_{bas}$.

That prop gives us a unique morphisms $T' \to T_{bas}$, $X' \to X_{bas}$ & a Cartesian diagram

$$\begin{array}{ccc} X' & \longrightarrow & X_{bas} \\ \downarrow & & \downarrow \\ T' & \longrightarrow & T_{bas} \end{array}$$

of logsch.

Now $\dfrac{X_{bas}}{T_{bas}}$ ~~comes~~ gives universal map $T_{bas} \to \mathcal{M}_{g,n}^{bas}$, which is what we wanted.

("□")